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MATHEMATICAL GAZETTE.

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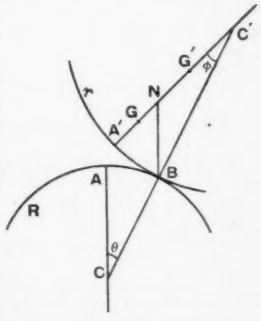
THE ELEMENTARY THEORY OF STATICAL STABILITY.

BY DR. S. BRODETSKY, UNIVERSITY OF BRISTOL.

THERE are some problems in elementary mathematics that the average student is apt to consider as matter for memorising without any great effort being either useful or advisable at understanding them. Experience in teaching elementary statics shows that the theory of stability is an example of such a problem. The average student often does not appreciate the principle of the method generally adopted to investigate the stability of the equilibrium of one body resting on another. He is not generally in a position to grasp the significance of an "infinitesimal displacement," or the meaning of "proceeding to the second order of small quantities" when the equilibrium is "neutral" to the first order. He is further puzzled by the fact that what appears to be neutral equilibrium becomes, when higher orders of small quantities are taken into account, stable in one case, unstable in another, whilst remaining neutral in a third. These difficulties can be overcome if we adopt a slight modification of the treatment given in the text-books. The object of this article is to explain the method used by the author for lecturing purposes, and not to obtain new results.

Let us take the elementary problem as ordinarily formulated. A body with a spherical base rests on a fixed body with a spherical top, sufficiently rough to prevent slipping when a displacement takes place: to investigate the nature of the equilibrium.



Let the figure represent a displacement of the top body. A is the original point of contact, C the centre of the lower sphere. B is the new point of contact, C' the new position of the centre of the top sphere, so that CBC' are collinear. $C'A'$ is the new position of that radius of the top sphere that was originally vertical, so that A' was originally at A. The centre of gravity of the top body is somewhere in the line $A'C'$, produced if necessary.

At B draw BN vertically to meet $C'A'$ in N. Following a notation suggested by Minchin, we shall call N the critical point for the displacement defined by the angle ACB $(=\theta)$. N will in general vary with θ. If the centre of gravity of the top body is below N, say at G, the gravity will bring the body back to its original position; if the centre of gravity is above N, say at G', the gravity will

take the top body still further away from its original position; if the centre of gravity is at N, neither effect will be produced.

Now suppose that when θ is zero, the critical point is at N_0, and let θ be gradually increased from zero. The critical point, being dependent on θ, will either begin to rise, or to sink: as a limiting case it may happen to be functionally independent of θ and remain at N_0. If N *rises from* N_0, then when the centre of gravity is *below or at* N_0, there will be a restoring couple in a disturbed position and the equilibrium is *stable*, but when the centre of gravity is *above* N_0, the equilibrium is *unstable*, since for a small enough value of θ, the centre of gravity is above N. If N *sinks from* N_0, then when the centre of gravity is *above or at* N_0, it remains above N and the equilibrium is *unstable*, but when the centre of gravity is *below* N_0, the equilibrium is *stable*, since for a small enough value of θ, the centre of gravity is below N. If N *remains at* N_0, then when the centre of gravity is *below* N_0 the equilibrium is *stable*, if *above* N_0 the equilibrium is *unstable*, but if the centre of gravity is *at* N_0 the equilibrium is truly *neutral*.

To apply these results, let us call $A'N \equiv H$, the critical height for the angle θ. H can be easily calculated in terms of θ. Let the angle through which the top body has rotated be called ϕ, and let the radius of the top sphere be r and of the bottom sphere R; then $r\phi = R\theta$, and, equating the horizontal distance of N from C' to that of B from C', we get

$$r\sin\theta = (r-H)\sin(\theta+\phi),$$

i.e.

$$H = r\left\{1 - \frac{\sin\theta}{\sin\dfrac{R+r}{r}\theta}\right\}. \quad \ldots\ldots(1)$$

We shall assume at present that neither body is flat, so that r and R are both finite. When θ is zero the critical height is

$$H_0 = \frac{Rr}{R+r}. \quad \ldots\ldots(2)$$

N will rise from N_0 as θ increases from zero if H increases with θ; N will sink if H decreases as θ increases from zero; N will remain at N_0 if H does not really involve θ.

The only case giving stationary N is when $\sin\dfrac{R+r}{r}\theta$ is a multiple of $\sin\theta$. For finite values of r, R this can only occur when $R = -2r$, and then $H = H_0 = 2r$. Thus, if the lower sphere is also concave upwards and its radius is double that of the top sphere, the equilibrium is truly neutral when the centre of gravity of the top body is at a height equal to the diameter of the top sphere; the equilibrium is stable if the centre of gravity is lower, unstable if higher.

To investigate the general case we use the following facts:

$$\frac{\sin n\theta}{\sin\theta},\ n>1,\ \text{decreases as } \theta \text{ increases from zero}, \quad \ldots\ldots(3)$$

$$\frac{\sin n\theta}{\sin\theta},\ n<1,\ \text{increases as } \theta \text{ increases from zero}. \quad \ldots\ldots(4)$$

The student can readily verify these statements either by differentiation if he is sufficiently advanced in his pure mathematics, or by means of plotting the function $\sin n\theta/\sin\theta$ for different values of n by means of a table of sines.

We now consider the three cases that can occur with finite radii. If the top sphere is concave upwards and the bottom sphere is concave downwards, then $(R+r)/r$ is greater than unity, and therefore by (3) H decreases as θ increases from zero, *i.e.* N sinks from N_0. Thus in this case the equilibrium is unstable if the centre of gravity of the top body is above or at the height $Rr/(R+r)$, stable if the centre of gravity is lower.

If both spheres are concave upwards, we use $-R$ instead of R for the bottom sphere, and we must write

$$\left.\begin{aligned} H&=r\left\{1+\frac{\sin\theta}{\sin\dfrac{R-r}{r}\theta}\right\},\\ H_0&=\frac{Rr}{R-r}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

R is of course greater than r. Here we get three sub-cases.

If $R>2r$, $\sin\dfrac{R-r}{r}\theta/\sin\theta$ decreases as θ increases from zero, and therefore by (5) H increases, so that N rises from N_0. Thus, if the centre of gravity of the top body is below or at the height $Rr/(R-r)$, the equilibrium is stable; if the centre of gravity is higher the equilibrium is unstable.

If $R=2r$, $H=H_0=2r$, and we get truly neutral equilibrium when the centre of gravity of the top body is at the height $2r$, stable when below, unstable when above.

If $R<2r$, $\sin\dfrac{R-r}{r}\theta/\sin\theta$ increases as θ increases from zero, by (4), and therefore by (5) H decreases as θ increases, so that N sinks from N_0. Thus, if the centre of gravity of the top body is above or at the height $Rr/(R-r)$ the equilibrium is unstable; if lower the equilibrium is stable.

When both spheres are concave downwards, we use $-r$ instead of r for the top sphere, and we get

$$\left.\begin{aligned} H&=r\left\{-1+\frac{\sin\theta}{\sin\dfrac{r-R}{r}\theta}\right\},\\ H_0&=Rr/(r-R). \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

r is greater than R, but $(r-R)/r$ is always less than unity. Hence $\sin\dfrac{r-R}{r}\theta/\sin\theta$ increases as θ increases from zero, and H decreases, so that N sinks from N_0. The equilibrium is therefore unstable if the centre of gravity of the top body is above or at the height $Rr/(r-R)$, stable if lower.

The cases when either r or R is infinite, *i.e.* when either the top or the bottom body is flat, need special investigation, but they are very simple. When R is infinite, then N is clearly at the centre of the top sphere and invariable. The equilibrium is truly neutral if the centre of gravity of the top body is at the centre of its sphere, stable if lower, unstable if higher.

When r is infinite, we get at once

$$\left.\begin{aligned} H&=R\theta/\tan\theta,\\ H_0&=R. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The student will readily verify that $\tan\theta/\theta$ increases as θ increases from zero, so that N sinks from N_0. The equilibrium is therefore unstable if the centre of gravity is above or at the height R, stable if lower.

In this manner the difficulty experienced by the student can be avoided, and at the same time we eliminate the use of the expansions for $\sin\theta$, $\sin\dfrac{R+r}{r}\theta$, etc., in terms of θ, expansions that come comparatively late in pure mathematics.

We can use the same method in the more general problem of the two-dimensional stability for any two surfaces, the top one movable but resting on the lower one which is fixed, there being sufficient friction to prevent slipping. We shall suppose that in equilibrium the centre of gravity of the top body is in the vertical common normal at the point of contact.

We can use the same figure as before, the curves being any curves, not necessarily circles. Let it represent a two-dimensional displacement. A is the original point of contact, B the new point of contact, AC is the vertical at C, $C'BC$ is the common normal at B meeting AC in C, and $A'C'$ in C', where A' is the point of the top body that was originally in contact with A and $A'C'$ the normal to the top body at A'. Draw BN vertically through B to meet $A'C'$ in N. Then N is the critical point for the displacement represented by AB, and will vary as AB increases from zero. Let N_0 be the critical point for AB zero. Then, if N rises from N_0 as AB increases from zero, the equilibrium is stable when the centre of gravity of the top body is below or at N_0, unstable if higher. If N sinks, the equilibrium is unstable when the centre of gravity is above or at N_0, stable if lower. If N remains permanently at N_0, the equilibrium is stable when the centre of gravity is below N_0, neutral when at N_0, unstable when above N_0.

To apply this principle, let $A'N \equiv H$. Then

$$H = y_2 + x_2 \cot(\theta + \phi), \qquad \text{.....(8)}$$

where θ is the angle ACB, ϕ is the angle $A'C'B$; x_1, y_1 are the coordinates of B, taken on the lower curve, referred to an origin at A, a y-axis vertically downwards and an x-axis horizontally to the right; x_2, y_2 are the coordinates of B, taken on the upper curve, referred to an origin at A', a y-axis along $A'C'$ and an x-axis along the tangent at A' to the right. (This is very similar to Minchin's treatment.) Putting

$$\left.\begin{aligned} \tan\phi = \frac{dy_2}{dx_2}, \quad \tan\theta = \frac{dy_1}{dx_1}, \\ H = y_2 + x_2 \frac{\dfrac{dx_1}{ds}\dfrac{dx_2}{ds} - \dfrac{dy_1}{ds}\dfrac{dy_2}{ds}}{\dfrac{dx_1}{ds}\dfrac{dy_2}{ds} + \dfrac{dy_1}{ds}\dfrac{dx_2}{ds}}, \end{aligned}\right\} \qquad \text{.....(9)}$$

we get

s being the distance AB or $A'B$ measured along either curve.

Using the expansions

$$\left.\begin{aligned} x_1 &= s - \frac{s^3}{6\rho_1^2} + \frac{s^4}{8\rho_1^3}\frac{d\rho_1}{ds} + \dots, \\ y_1 &= \frac{s^2}{2\rho_1} - \frac{s^3}{6\rho_1^2}\frac{d\rho_1}{ds} + \dots, \end{aligned}\right\} \qquad \text{.....(10)}$$

where ρ_1 is the radius of curvature for the lower curve, with similar equations for the upper curve in terms of its radius of curvature ρ_2, we find

$$H = \frac{\rho_1\rho_2}{\rho_1+\rho_2} + C_1 s + C_2 s^2 + \dots, \qquad \text{.....(11)}$$

where C_1, C_2 are numbers depending on ρ_1, ρ_2 and their variations at A and A'.

Hence

$$H_0 = \frac{\rho_1\rho_2}{\rho_1+\rho_2}. \qquad \text{.....(12)}$$

N will rise or sink from N_0 according as dH/ds is positive or negative near the original point of contact, *i.e.* according as C_m is positive or negative, where C_m is the first coefficient in $H - H_0$ which does not vanish.

Thus the equilibrium if stable for the centre of gravity of the top body at the height $\rho_1\rho_2/(\rho_1+\rho_2)$ if C_m is positive, unstable if C_m is negative and truly neutral if no such coefficient exists at all, so that $H \equiv H_0$. In all cases the equilibrium is stable when the centre of gravity is lower than $\rho_1\rho_2/(\rho_1+\rho_2)$, unstable if higher.

In this way the whole problem of statical stability can be discussed in an elementary manner without the introduction of anything more difficult than the increase or decrease of a certain function of a given variable. It is also evident that by plotting the values of H given by (1), (5), (6), (7), (9) the problem of a *finite* displacement admits of easy discussion. S. BRODETSKY.

ALICE THROUGH THE (CONVEX) LOOKING GLASS.*

The Presidential Address to the London Branch of the Mathematical Association, February 9th, 1918.

By Wm. Garnett.

Oh, wad some power the giftie gi'e us
To see oursels as others see us,
It wad frae mony a blunder free us,
And foolish notion ;

in which case it is probable that wars would cease. During the last few years we have had glimpses of ourselves through the spectacles of the Central Powers ; occasionally through the telescopes of the French, the Italians, or the Russians, and in the near future we are likely to get a good many pictures of ourselves as we appear to the last-named people. We are apt to think that our own view of ourselves is the only true view, and all other images are in error just as they differ from our own standard. Naturally other nations adopt the same course, and we are free to believe that in many cases their vision of themselves is far more at fault than our own self image. In this respect a study of Alice and her standards might be not without profit to the nations. When we want to see ourselves, or, at least, our faces, we usually employ a mirror, and what we see depends very much on the mirror we use. Sir George Greenhill was the first professor of Applied Mathematics at the Royal Indian Civil Engineering College, Coopers' Hill. He told us that at the end of the first session a reception was held and the science departments were on view. A young lady entering the physical laboratory and seeing an inverted image of herself in a large concave mirror naïvely remarked to her companion, "They have hung that looking glass upside down." But those who fixed the mirror were not to blame, as the lady would have discovered had she advanced past the centre of the curvature. The reply of modern mathematics to the prayer of Robert Burns is the Theory of Relativity. I have some doubt whether Burns would have appreciated the answer. As a rule, we use a plane mirror to inspect ourselves. This reverses certain spatial relations which can be expressed best, perhaps, by stating that it presents a right-handed screw as a left-handed screw and *vice versâ*, or changes a wine screw into a beer screw, as Clerk Maxwell would say, referring to the tendrils of the vine and the hop.

In my undergraduate days I was very fond of Alice. I wanted someone to write the Adventures of Alice through a Convex Looking Glass. Hitherto I have not seen any literature on this subject ; but in order to throw some light on what might be involved in a realisation of Burns' desire, and in the hope that some member of the Association will deal seriously with the mathematics of the situation, I shall venture to call your attention to a few of the simpler issues.

Let us suppose that the real Alice has passed through the mirror and is living in Convex Looking-Glass Land. As we want to give her plenty of room to move freely and to play with her toys, we shall assume that the radius of the mirror is very large as compared with the height of Alice before she passed through the glass. It must not be supposed that we are entirely altruistic in this liberality of space. It obviates the necessity of

* This "Presidential Address" was written for the purpose of providing an hour's relaxation, and not as a contribution to Mathematical Science or with any view to publication. If it has any merits, they are to be found exclusively from the point of view of Elementary Education in so far as it offers an easy introduction to the conception of variable units of space and mass, and so prepares the mind for the Theory of Relativity.

dealing with the curvature of images and with other details which greatly complicate the mathematics; so, by giving Alice plenty of room for her play, we save ourselves the trouble of dealing with formidable calculations and leave plenty of new ground for future explorers in this space. We will further suppose that an imaginary Alice of approximately constant dimensions is disporting herself in front of the mirror in Euclidean space and is obeying the Newtonian laws of Dynamics. I want us to identify ourselves at first with the life of the real Alice in Looking-Glass Land and see how she behaves, it being understood that in all her actions she corresponds to the imaginary Alice outside. I have called her the real Alice because, for the present, her life is to be our centre of interest. I will suppose our Alice to be of a scientific turn of mind and, while not despising toys, to have provided herself with a measuring rule for length, a clock or other means of measuring time, and some means of measuring mass by comparison with a standard unit.

If the imaginary Alice move away from the mirror with uniform velocity, the real Alice will also move away from the mirror in the opposite direction, but she will never traverse half the radius so as to reach the principal focus, F. In fact, F is Alice's infinity, and everything that she regards as infinitely distant lies on a sphere concentric with the mirror and of half its radius. All lines parallel to the axis and within Alice's reach (Alice is small compared with the radius) meet at F, but don't let us assume prematurely that Euclid's fifth postulate (or twelfth axiom) is not true of these lines in Alice's experience. To an observer outside the mirror these lines appear to meet in a point at a finite distance, and as Alice approaches this focus she seems to get smaller and smaller and to move more and more slowly, but can she or can we, if we live in the same space with her, be conscious of this change? Two lengths are equal if their extremities can be made simultaneously to coincide, and Alice carries with her a standard measure. When this measure is applied to her body its length has diminished in exactly the same ratio as the dimensions of Alice which it is employed to measure. Consequently, if Alice's height were 4′ 6″ before she passed through the glass she will continue to measure 4′ 6″ by her own measuring standard however near she may get to the focus. If any of you have not accompanied Alice and myself into Convex Looking-Glass Land you will be foolish enough to think that Alice's height will be directly proportional to her distance from the principal focus as you see it. So it is to Alice, but to her the principal focus is at infinity. It is the point of meeting of parallel lines and Alice remains between the same parallels, and therefore her height is constant. It is clear that all this will be equally true of any of her dimensions measured at right angles to the axis of the mirror. If we adopt the usual symbols, u, v, f and d, for the distances from the vertex of the mirror of the imaginary Alice, the real Alice and the principal focus and for the distance of the real Alice from the principal focus, then Alice's dimensions perpendicular to the axis of the mirror vary directly as d and u, v, f and d are connected by the relations

$$\frac{1}{u}+\frac{1}{v}=-\frac{1}{f} \quad \text{.....(1)} \quad \text{and} \quad \frac{d}{f}=\frac{f}{u+f} \quad \text{.....(2)}$$

But what about the dimensions of Alice parallel to the axis of the mirror? Alas, in this direction Alice will appear to the outsider to suffer more than in the direction just considered, but she will show no signs of anxiety, and her physicians will be unable to detect any undesirable symptoms. To determine Alice's dimensions from front to back equation (1) gives

$$v=-\frac{uf}{u+f}.$$

If we take the small quantity δu to represent one of the dimensions parallel to the axis of the imaginary Alice, and therefore of the real Alice before she

passed through the mirror, and δv the corresponding dimensions of the real Alice at the distance v from the mirror, then

$$\delta v = -\frac{f^2}{(u+f)^2}\delta u, \quad = -\frac{d^2}{f^2}\delta u \text{ by (2)},$$

the minus sign merely indicating that the dimension is measured in the opposite direction, that is, that Alice is perverted. Thus to the outsider Alice's dimensions parallel to the axis of the mirror appear to vary as the square of her distance from the principal focus, and therefore in the duplicate ratio of her dimensions perpendicular to the axis.

Alice in the outer world. Alice perverted and reduced to half-scale. Alice as she appears midway between the mirror and the principal focus.

FIG. 1.

But when Alice turns her foot rule or metre rod in the direction of the axis, it too suffers a corresponding change of length as judged by the outsider, though Alice has no means of detecting this, and consequently, gauged by her own measurements, her dimensions remain unchanged. Let us, however, remember that, to the outsider, Alice's unit of length is reduced in the ratio $\frac{d}{f}\left(\text{or } \frac{f}{f+u}\right)$ when it is held perpendicularly to the axis, and in the ratio $\frac{d^2}{f^2}\left(\text{or } \frac{f^2}{(u+f)^2}\right)$ when it is parallel to the axis.

If Alice holds up a toy windmill, the vanes of which spin round as Alice moves forward, if the plane of rotation is perpendicular to the axis the space traversed by the rotating vanes will appear circular to the outsider as well as to Alice as it advances, though to the outsider its diameter will appear to diminish, but if Alice trundles a hoop, to the outsider it will assume the form of an ellipse with its vertical axis $\frac{f}{d}$ times its horizontal axis. As measured by Alice's standards all the diameters will appear of the same length, and the hoop will remain circular.

Now let Alice put aside her hoop and spin a top. We will assume that the corresponding top of the imaginary Alice is truly circular and spins with its axis vertical, *i.e.* at right angles to the axis of the mirror. The outsider looking down on it will see the strange phenomenon of an elliptic top with axes in the ratio of f to d spinning while the elliptic outline remains fixed in space, and every particle of the top describes its appropriate ellipse; in fact, in Convex Looking-Glass Land, as judged by an outsider, a rigid body

changes its dimensions in the ratio of f to d twice during every revolution in a plane containing the axis of the mirror.

Some persons may, perhaps, be inclined to go outside the domain of geometry, or even mechanics, and ask whether Alice could not detect the ellipticity of the top by looking at it. Visual impressions are interpreted and correlated with the sense of touch only by experience. Without experience we may "see men as trees walking." If Alice hold the top before her with its axis parallel to that of the mirror and look straight at it, the top, Alice's eyes and the image of the top on Alice's retina will all appear to the outsider to be circular, but when Alice looks down at the spinning top, the top, Alice's eyes and the retinal images will all appear to the outsider to be similarly elliptic, but the image will cover exactly the same area of the retina as before, if the distance be properly adjusted, and therefore Alice's perception of the circle will remain unchanged. The same remarks will apply to Alice's estimate of the shape of her hoop when she looks at it in a direction perpendicular to the axis of the mirror.

So far then Alice appears to be entirely unconscious of the extraordinary things which appear to the outsider to be happening to her and to everything which is moving in her world. Now let us for a moment consider Alice's system of mechanics.

Let the imaginary Alice walk away from the mirror with uniform velocity V_1, so that $\frac{du}{dt} = V_1$. Then, since $v = -\frac{uf}{u+f}$, it follows that

$$\frac{dv}{dt} = -\frac{f^2}{(u+f)^2}\frac{du}{dt}, \quad V_1 = -\frac{d^2}{f^2}V_1,$$

so that to the outside observer the real Alice appears to walk towards the principal focus at a rate diminishing as the square of her distance from the focus, or inversely as the square of the distance of the imaginary Alice from F. But Alice's standard of length, when used to measure distances in the direction of the axis varies in the same ratio. Hence, if Alice's unit of time is the same as that of the imaginary Alice who obeys the Newtonian laws, her measure of her own velocity will be constant and equal to V_1.

It may be that Alice's unit of time is not quite the same as that of the imaginary Alice, but it will differ only by a quantity proportional to $\left(\frac{\frac{du}{dt} - \frac{dv}{dt}}{c}\right)^2$, where c is the velocity of light. This is a small quantity of a very high order from the point of view of our approximations, and we shall therefore suppose that Alice adopts the same unit of time as her imaginary co-partner.

Alice will now amuse herself with a ball. We suppose the ball to be thrown vertically upwards, and therefore at right angles to the axis of the mirror. The imaginary Alice, of course, throws up an imaginary ball which behaves as every properly conducted ball should. Let V_t denote the velocity of this ball at any instant. Then in the small time δt the imaginary ball moves through the distance $V_t\,\delta t$ perpendicular to the mirror axis. The length of the image of this distance in the mirror is $\frac{d}{f}V_t\,\delta t$, and this is the distance which the real ball thrown up by the real Alice appears to the outside observer to traverse in the same time. His estimate, therefore, of the velocity of the real ball is $\frac{d}{f}V_t$. But his estimate of Alice's unit of length is reduced in the ratio of $\frac{d}{f}$, and as Alice measures the velocity of the ball by this standard her estimate of the velocity will be V_t, and the ball will appear to her to move in accordance with the same law as the imaginary ball in Euclidean space.

What is true of velocity will necessarily be true of change of velocity and, with the same unit of time, of rate of change of velocity, or acceleration. Hence Alice's measures of these quantities will be the same as the measures of the corresponding quantities in the outside world.

So far we have dealt only with the geometry of space and with kinematics. What will be Alice's views about Dynamics? We are probably justified in assuming the principles of the Conservation of Momentum and the Conservation of Energy. Subject to no forces the imaginary Alice, if in motion, will continue to move uniformly in a straight line. Let her be moving parallel to the axis with velocity V_t. Then the real Alice will be moving in the opposite direction with velocity V_t', where V_t' is equal to $\frac{d'^2}{f^2}V_t$ as estimated by the external observer. If M_t' denote the mass of the real Alice as measured by the external observer, then $M_t'V_t'$ must be constant while V_t' appears to vary as d'^2. Hence, to the external observer M_t' must vary inversely as d'^2, or the mass of Alice, as estimated by the external observer in terms of his own unit of mass, must vary inversely as the square of the distance of Alice from the principal focus or directly as the square of the distance of the imaginary Alice from the same point. It will be remembered that Alice's linear dimensions appear to the external observer to vary as d, d and d^2 respectively, and therefore her volume varies as d^4. Hence the density of the real Alice estimated by the external observer in relation to motion along the axis varies inversely as d'^6, or as the inverse sixth power of the distance of the imaginary Alice from the principal focus, a somewhat surprising result.

If we consider the movements of Alice at right angles to the axis of the mirror, using the same notation as before, we have $V_t' = \frac{d}{f}V_t$, and the condition that the momentum may remain constant is that $M_t'V_t'$, or $M_t'\frac{d}{f}V_t$, should be constant. Therefore M_t' as estimated by the external observer varies inversely as d, or the mass of the real Alice estimated by the external observer in relation to motion perpendicular to the axis varies inversely as her distance from the principal focus and directly as the distance of the imaginary Alice from the same point. Alice's density similarly measured varies inversely as the fifth power of her distance from F.

Members of my audience who accept the modern Theory of Relativity will have no difficulty in recognising that Alice's mass may vary with her position and also with the direction of motion with respect to which it is measured. It will be remembered that, according to that theory, the mass of a body increases indefinitely with its velocity relative to the observer, becoming infinite when that velocity is equal to the velocity of light, and that the mass is different when estimated in relation to motion along and perpendicular to the relative velocity with respect to the observer.

But during all these changes Alice herself is quite unconscious of any alteration in her mass or density. To the external observer Alice's standard pound or kilogramme will appear to change in exactly the same way as the mass of Alice herself with change of position or change of direction of motion. Hence Alice's measure of her own mass or of that of any other body in her world will be independent of position or direction of motion. We have seen that Alice's measure of velocity is constant as long as the velocity corresponds to uniform velocity in the external world. Hence, under the same circumstances, Alice's measure of momentum remains constant, and is equal to the measure of the corresponding momentum made by the imaginary Alice, so that the conservation of momentum places her in no difficulty.

(*To be continued.*)

THE INTRODUCTION TO INFINITE SERIES.*

By W. J. Dobbs, M.A.

In the Report of the Education Reform Council, published last year under the title "Education Reform," a paragraph occurs on page 84 with reference to the Mathematics Course suitable for boys of 16 to 18 years of age attending a Secondary School. There it is suggested that such boys who take up Mathematics and Science may pursue the study of differential and integral calculus, including the easy parts of differential equations, and apply these with advantage to higher algebra, trigonometry, coordinate geometry, solid geometry and mechanics; that the course should include some introduction to infinite series; that too much time has been devoted hitherto to the binomial series; and that the thing to aim at is Taylor's series.

The school course in Higher Mathematics with which most of us are familiar involves much work of little educational value on the binomial series, including the extension from a positive integral index to other indices, *i.e.* from a finite to an infinite series, as given, for instance, in Todhunter's *Algebra*. Then the exponential, logarithmic, and trigonometrical series are based upon the binomial series. Finally, a very few schoolboys are introduced to Taylor's series by assuming the existence and validity of an expansion in ascending integral powers of the variable. Any treatment of infinite series suitable for the ordinary schoolboy is hampered by the difficulties inseparable from considerations of convergency. Yet it seems desirable that schoolboys should have some acquaintance with the most important of these series and be able to apply them to practical use. The problem is an important one, but a very thorny one, and I hope that some mathematician familiar with the limitations of the schoolroom, but more able to deal with it than I can claim to be, will take the matter up. The treatment should be such that the boy will have nothing to unlearn, and that he may acquire such knowledge as he can put to practical use. It is not necessary that the course should be completely rigorous from beginning to end, but it is desirable that all the steps should be capable of a rigorous justification perhaps at a later stage. It may not infrequently happen that an assumption is made which the boy may not at the time be able to justify; but such assumptions should be honestly faced and admitted. And we must never argue about an infinite number of quantities as we would about a finite number.

In a recently published school algebra† by a well-known writer who formerly thought differently, the binomial, exponential and logarithmic infinite series are now merely stated without proof, and their uses illustrated. Perhaps this will turn out to be the only practicable method for school work, but I venture to hope that a better method may be found. I wish to place before this association a few suggestions for consideration. If they do not stand the test of searching criticism, they may perhaps stimulate others to attack the problem with greater success.

The early introduction of calculus methods into school work, if developed along suitable lines, provides a quicker, and perhaps a better, means of access to these series than was possible when the vast majority of schoolboys were denied access to the elements of infinitesimal calculus. Clearly, if the calculus method is to be available, the rules for differentiating the standard functions must not be based on the accepted expansions of those functions in infinite series.

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* Read at a meeting of the London branch on the 9th of March, 1918.

† *A School Algebra*, by H. S. Hall.

The preliminary work also involves a discussion of the properties of the function $y=a^x$, where a is a positive quantity. This arises in connection with the theory of indices and logarithms. The addition of $\frac{1}{n}$ to the value of x applies the growth-factor $\sqrt[n]{a}$ to the value of y, n being a positive integer. If $a>1$, y increases without limit as $x\to\infty$, and y, remaining positive, can be made $<\epsilon$ by sufficiently increasing numerically the negative value of x, ϵ denoting a quantity to which any given positive value may be assigned as small as we please. If $a<1$, y, remaining positive, can be made $<\epsilon$ by sufficiently increasing the positive value of x, and y increases without limit as $x\to-\infty$. The gradient of the graph is proportional to y and becomes identical with y when a has the special value e. This association has already suffered at my hands a paper on this subject a few years ago.*

Let us suppose then that, instead of spending much time upon the binomial series and other more or less undesirable topics, the schoolboy has learned how to differentiate such functions as x^a, a^x, $\log x$, $\sin x$, $\cos x$, $\tan^{-1}x$. Also, in connection with the consideration of irrational numbers, he has been introduced to the idea of a sequence. Recently Professor Carey's *Infinitesimal Calculus* has presented this topic in a very attractive form, and I heartily commend his book to your notice.

We will consider in turn a few of the most important infinite series. I shall throughout use the symbol n to denote a positive integer and the symbol ϵ to denote a quantity to which any given positive value may be assigned as small as we please.

(i) *Expansions of $\sin x$ and $\cos x$.* We will take as our first examples the expansions of $\sin x$ and $\cos x$ in ascending powers of x. The schoolboy requires these expansions only for values of x ranging from 0 to $\frac{1}{4}\pi$, *i.e.* for positive angles less than 45°. We will, however, limit ourselves to values of x ranging between 0 and 1, *i.e.* to angles ranging between 0 and about 57·3°.

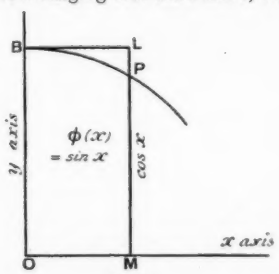


FIG. 1.

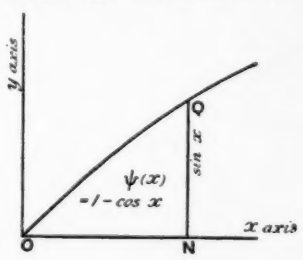


FIG. 2.

In the left-hand figure, which represents the graph of $y=\cos x$ for values of x ranging between 0 and 1, let $\phi(x)$ denote the area $OMPB$ between the axes of coordinates, any ordinate MP and the curve. In the right-hand figure, which represents the graph of $y=\sin x$ for values of x ranging between 0 and 1, let $\psi(x)$ denote the area ONQ between the x axis, any ordinate NQ and the curve.

Then $\quad \phi'(x)=\cos x \quad$ and $\quad \phi(0)=0\,;$

$$\therefore\ \phi(x)=\sin x.$$

Also $\quad \psi'(x)=\sin x \quad$ and $\quad \psi(0)=0\,;$

$$\therefore\ \psi(x)=1-\cos x.$$

* See *Mathematical Gazette*, vol. v. p. 179.

Now, except when $x=0$, $\cos x<1$;

$\therefore$ area $OMPB<$ area $OMLB$,

i.e. $\sin x<x$.

Thus

$$\begin{aligned}
&\sin x>0 \quad \text{but} \quad <x;\\
\therefore\ &\psi(x)>0 \quad \text{but} \quad <\tfrac{1}{2}x^2,\\
\textit{i.e.}\ &1-\cos x>0 \quad \text{but} \quad <\tfrac{1}{2}x^2;\\
\therefore\ &\cos x<1 \quad \text{but} \quad >1-\tfrac{1}{2}x^2;\\
\therefore\ &\phi(x)<x \quad \text{but} \quad >x-\frac{x^3}{\lfloor 3},\\
\textit{i.e.}\ &\sin x<x \quad \text{but} \quad >x-\frac{x^3}{\lfloor 3};\\
\therefore\ &\psi(x)<\tfrac{1}{2}x^2 \quad \text{but} \quad >\frac{x^2}{\lfloor 2}-\frac{x^4}{\lfloor 4},\\
\textit{i.e.}\ &1-\cos x<\frac{x^2}{\lfloor 2} \quad \text{but} \quad >\frac{x^2}{\lfloor 2}-\frac{x^4}{\lfloor 4};\\
\therefore\ &\cos x>1-\frac{x^2}{\lfloor 2} \quad \text{but} \quad <1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4};\\
\therefore\ &\phi(x)>x-\frac{x^3}{\lfloor 3} \quad \text{but} \quad <x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5},\\
\textit{i.e.}\ &\sin x>x-\frac{x^3}{\lfloor 3} \quad \text{but} \quad <x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5},
\end{aligned}$$

and so on.

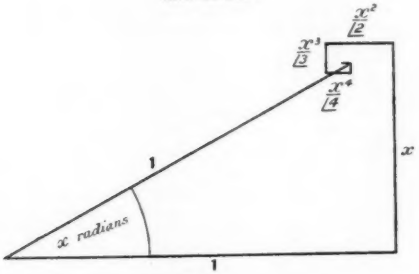


FIG. 3.

Thus we have the sequences of functions

$$\left\{\begin{array}{l} 1,\\ 1-\dfrac{x^2}{\lfloor 2},\\ 1-\dfrac{x^2}{\lfloor 2}+\dfrac{x^4}{\lfloor 4},\\ \text{etc.}\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l} x,\\ x-\dfrac{x^3}{\lfloor 3},\\ x-\dfrac{x^3}{\lfloor 3}+\dfrac{x^5}{\lfloor 5},\\ \text{etc.}\end{array}\right\}$$

the former giving successive approximations to $\cos x$ alternately too great and too small, the latter giving successive approximations to $\sin x$ alternately too great and too small. In both series every new term introduced is smaller than that which preceded it, and each term is of the form $\pm\dfrac{x^n}{\lfloor n}$, the

numerical value of which can be made $<\epsilon$ by sufficiently increasing n. Here we have two ideal series for purposes of calculation, and can at every stage assign an outside estimate to the error remaining after a finite number of terms has been used. Though x may be limited to values ranging from 0 to $\frac{1}{4}\pi$, we have here suitable machinery for the calculation, to any required degree of accuracy, of any trigonometrical function of any angle. These expansions are therefore particularly suitable for school use.* The figure on p. 244 is an obvious geometrical illustration.

(ii) *Geometric Progression.* Secondly, let us take the familiar series whose general term, the $(n+1)$th, is x^n, and, denoting $\frac{1}{1-x}$ by f, consider the sequence

$$\phi_1 = 1+x = f-x^2 f,$$
$$\phi_2 = 1+x+x^2 = f-x^3 f,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\phi_n = 1+x+x^2+\dots+x^n = f-x^{n+1}f.$$

If x is numerically less than p, where p is a positive quantity less than 1, x^n can be made numerically $<\epsilon$ by sufficiently increasing n, and f remains finite, though it varies with x between the limits $\frac{1}{1+p}$ and $\frac{1}{1-p}$. Thus the limit of the sequence is f for all values of x ranging from $-p$ to $+p$. In writing f as an approximation to the sum of a finite number n of terms of the series, we know that the error is $x^n f$ exactly.

A very beautiful and illuminating geometrical illustration of this convergence is given in Carslaw's *Plane Trigonometry*. Take a line-segment OA of unit length, and draw OP of gradient x and AP of gradient 1, meeting at

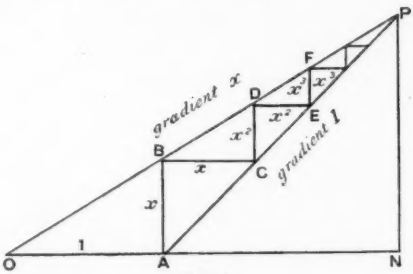


FIG. 4.

P. Then OA, AB, BC, CD, DE, EF, etc., drawn successively at right angles, have the values 1, x, x, x^2, x^2, x^3, etc. Draw PN perpendicular upon OA produced. Then ON is the limit of the sum $1+x+x^2+\dots$, and denoting this by s, $NP = s-1$. Clearly $s-1 = xs$; whence $s = \frac{1}{1-x} = f$. The remainder $x^n f$ may also be obtained directly from the figure.

(iii) *Logarithmic Series.* Knowing that

$$\frac{d\log(1+x)}{dx} = \frac{1}{1+x},$$

* On this point Professor Nunn has anticipated me. Since this paper was written I have found the expansions of the sine and cosine treated in the same way on pages 308 and 309 of his *Exercises in Algebra*, Part II.

the preceding expansion suggests that, provided x is numerically less than 1, we may, with adequate reservations, write

$$\frac{d\log(1+x)}{dx}=1-x+x^2+\dots.$$

May we integrate, and, omitting the constant of integration because $\log(1+x)=0$ when $x=0$, say

$$\log(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\dots?$$

This step requires justification and elucidation.

Consider the sequence of functions

$$\phi_1=\log(1+x)-x,$$
$$\phi_2=\log(1+x)-x+\tfrac{1}{2}x^2,$$
$$\phi_3=\log(1+x)-x+\tfrac{1}{2}x^2-\tfrac{1}{3}x^3,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\phi_n=\log(1+x)-x+\tfrac{1}{2}x^2-\dots+(-1)^n\frac{1}{n}x^n.$$

We notice that these functions have a common value 0 when x is 0.

Denoting differentiation with respect to x by a dash, we have

$$\phi_n'=\frac{1}{1+x}-1+x-x^2+\dots+(-1)^nx^{n-1}$$
$$=(-1)^nx^n\left(\frac{1}{1+x}\right),\quad\text{summing the G.P.}$$

Thus, corresponding to the sequence of ϕ's, we have the following sequence of functions expressing the gradient-laws of the former sequence:

$$\phi_1'=-x\left(\frac{1}{1+x}\right),$$
$$\phi_2'=+x^2\left(\frac{1}{1+x}\right),$$
$$\phi_3'=-x^3\left(\frac{1}{1+x}\right),$$
$$\dots\dots\dots\dots\dots\dots$$
$$\phi_n'=(-1)^nx^n\left(\frac{1}{1+x}\right).$$

We notice that each of these functions is 0 when x is 0.

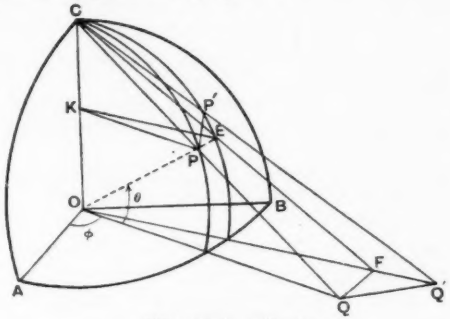
Now, when x ranges in value between $-p$ and $+p$, where p is a positive quantity less than 1, the numerical value of x^n can always be made as small as we please by sufficiently increasing n; at the same time $\frac{1}{1+x}$ remains finite, though it varies with x between the limits $\frac{1}{1-p}$ and $\frac{1}{1+p}$. Hence, for all values of x ranging from $-p$ to $+p$, ϕ_n' can be made numerically less than ϵ by sufficiently increasing n. For obvious reasons I have avoided values of x in the immediate neighbourhood of ±1. Indeed, for school purposes, it would be sufficient to allow x to range from $-\frac{1}{2}$ and $+\frac{1}{2}$.

Now, the graphs of ϕ_1, ϕ_2, ϕ_3, ... all touch the x axis at the origin, and their gradients all $\to0$ throughout the range $-p$ to $+p$ of x when $n\to\infty$. Thus it appears that for values of x ranging from $-p$ to $+p$ all the graphs tend towards the x axis as the limit when $n\to\infty$. May we infer that within this range $\phi_n\to0$ when $n\to\infty$, *i.e.* that $\log(1+x)$ is the limit of the sum $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\dots$ when the number of terms is indefinitely increased? We will consider this point a little more closely; and, as a similar digression will be required in connection with other series, we will take a more general case than the present one. *(To be continued.)*

MATHEMATICAL NOTES.

531. [K¹. 17. a, b.] *Integrations for a Rhumb Line.*

Let $OABC$ be an octant of a sphere whose centre is O and radius unity; PEP' an elementary triangle whose sides are differentials of distance, latitude, and dep.; QFQ' the projection from C on the plane OAB (hence stereographic); K the centre of the parallel of latitude θ, of which the arc PE is part. Then triangles KPE, OQF are similar, and ultimately, as may be shown easily, the triangles PEP', QFQ'.



Hence
$$\frac{FQ'}{OQ}=\frac{EP'}{KP}=\frac{EP\cot a}{KP},$$

i.e.
$$\frac{dOQ}{OQ}=\frac{d\theta}{\cos\theta}=d\phi\cot a.$$

Hence
$$\log\frac{OQ_2}{OQ_1}=\int_{\theta_1}^{\theta_2}\sec\theta\,d\theta=(\phi_2-\phi_1)\cot a,$$

where a is the angle of the course. E. M. Langley.

532. [V. 9.] *Note on a Singular Historical Error.*

On page 125, and elsewhere, of Eisenlohr's *Handbuch der alten Aegypter*, 1877, there appears the word *Ruthen* as a translation of an Egyptian word for a measure of length. On page 9 of the same work we are told that the length of this measure is unknown, so that the German word *Ruthen* = *Ruten* = rods is perhaps used here in a non-mathematical sense just as our word rods is commonly used in a non-mathematical sense.

In the *History of Greek Mathematics*, by James Gow, 1884, page 127, we find a reference to the same problem in which Eisenlohr had used the word *Ruthen*, and Gow uses *ruths* instead of the German *Ruthen*, while the Egyptian word was entirely different, and seems to have denoted pieces of wood (cf. Eisenlohr, page 128). In view of this meaning it would appear natural to have used in the English translation the term rods, which is the equivalent of the German word *Ruthen*.

It is a singular fact that the term *ruths* has found its way into a number of American books. In particular, it appears on page 44 of Cajori's *History of Elementary Mathematics*, 1896, as well as in the revised and enlarged edition of this work which was published in 1917. It appears also in some of our elementary text-books on geometry. For instance, in the 1916 edition of the Stone-Millis *Geometry*. As this error has apparently come into our American books from an English source, and as it is especially distasteful at the present time, it may possibly be of interest to direct attention to it in order to avoid its spreading still further. G. A. Miller.

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Gazette No. 8 (very important).
A.I.G.T. Report No. 11 (very important).
A.I.G.T. Reports, Nos. 10, 12.

GLEANINGS FAR AND NEAR.

21. The wide range of his (Schwarzschild's) contributions to knowledge suggests a comparison with Poincaré ; but Schwarzschild's bent was more practical, and he delighted as much in the design of instrumental methods as in the triumphs of analysis. It is interesting, though sometimes misleading, to trace a unity in the varied researches of a man's lifetime, for it often happens that a thread of association binds the whole together. But Schwarzschild's versatility baffles any such summary ; his joy was to range unrestricted over the pastures of knowledge, and, like a guerilla leader, his attacks fell where they were least expected. He sought no unity save the one bond which unites together all branches of scientific endeavour. "Mathematics, physics, chemistry, astronomy, march in one front. Whichever lags behind is drawn after. Whichever hastens ahead helps on the others. The closest solidarity exists between astronomy and the whole circle of exact science. I have often been untrue to the heavens. That is an impulse to the universal which was strengthened unwittingly by my teacher Seeliger, and afterwards was further nourished by Felix Klein and the whole scientific circle at Göttingen. There the motto runs that mathematics, physics, and astronomy constitute one knowledge, which, like the Greek culture, is only to be comprehended as a perfect whole." (Admission speech before the *Berlin Academy of Science*, 1913).—Prof. A. S. Eddington, *Monthly Notices* of the R.A.S., Feb. 1917.

22. "Something was done by way of preparing me for college by getting a Cambridge man, Bainbridge of Catherine Hall, to read Mathematics with me. I must have had some capacity, as he told my father when he left at the end of two months that if I were to go to Cambridge I should be certain to be a wrangler. But he was most likely a bad judge, as he did not himself get more than a Junior Optime."—Mark Pattison's *Memoirs*, p. 39.

[Bainbridge was bracketed second Jun. Op. 1831.]

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